The Filter Phase Function

Recall that the power transmission coefficient $\mathcal{T}(\omega)$ can be determined from the **scattering parameter** $S_{21}(\omega)$:

$$\mathcal{T}(\omega) = \left|S_{21}(\omega)\right|^2$$

Q: *I see, we only care about the* ***magnitude*** *of complex function $S_{21}(\omega)$ when using microwave filters !?*

A: Hardly! Since $S_{21}(\omega)$ is complex, it can be expressed in terms of its magnitude and **phase**:

$$\begin{aligned} S_{21}(\omega) &= \text{Re}\{S_{21}(\omega)\} + j\,\text{Im}\{S_{21}(\omega)\} \\ &= \left|S_{21}(\omega)\right| e^{j\angle S_{21}(\omega)} \end{aligned}$$

where the phase is denoted as $\angle S_{21}(\omega)$:

$$\angle S_{21}(\omega) = \tan^{-1}\left[\frac{\text{Im}\{S_{21}(\omega)\}}{\text{Re}\{S_{21}(\omega)\}}\right]$$

We likewise care **very** much about this phase function!

Q: *Just what does this phase tell us?*

A: It describes the relative phase **between** the wave incident on the input to the filter, and the wave exiting the output of the filter (given the output port is matched).

In other words, if the **incident** wave is:

$$V_1^+(z_1) = V_{01}^+ \; e^{-j\beta z}$$

Then the exiting (output) wave will be:

$$\begin{aligned} V_2^-(z_2) &= V_{02}^- \; e^{+j\beta z_2} \\ &= S_{21} \, V_{01}^- \; e^{+j\beta z_2} \\ &= \left|S_{21}\right| V_{01}^- \; e^{+j(\beta z + \angle S_{21})} \end{aligned}$$

We say that there has been a "**phase shift**" of $\angle S_{21}$ between the input and output waves.

Q: *What **causes** this phase shift?*

A: Propagation **delay.** It takes some non-zero amount of **time** for signal energy to propagate from the input of the filter to the output.

Q: *Can we tell from $\angle S_{21}(\omega)$ how **long** this delay is?*

A: Yes!

To see how, consider an **example** two-port network with the impulse response:

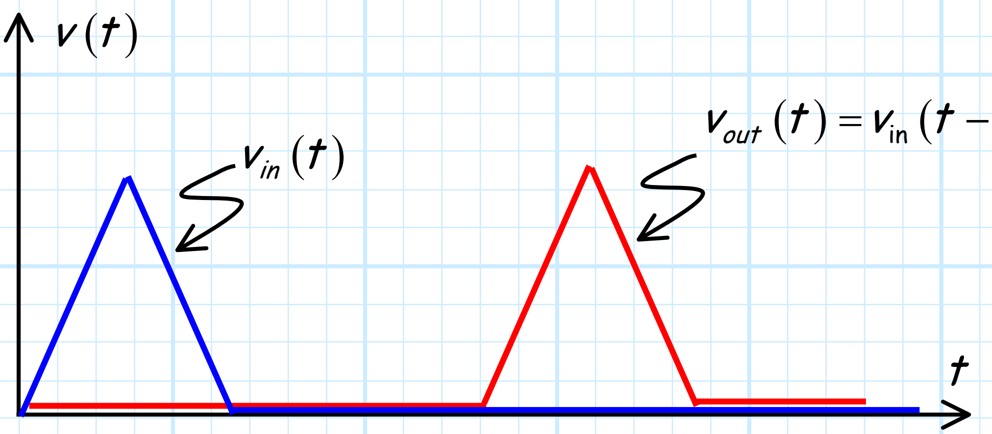
$$h(t) = \delta(t - \tau)$$

We determined earlier that this device would merely **delay** and input signal by some amount τ:

$$v_{out}(t) = \int_{-\infty}^{\infty} h(t - t')\, v_{in}(t')\, dt'$$

$$= \int_{-\infty}^{\infty} \delta(t - t' - \tau) v_{in}(t')\, dt'$$

$$= v_{in}(t' - \tau)$$



Taking the **Fourier transform** of this impulse response, we find the **frequency response** of this two-port network is:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t - \tau)\, e^{-j\omega t} dt$$

$$= e^{-j\omega\tau}$$

In other words:

$$|H(\omega)| = 1 \qquad \text{and} \qquad \angle H(\omega) = -\omega\tau$$

The interesting result here is the **phase** $\angle H(\omega)$. The result means that a delay of τ seconds results in an output "phase shift" of $-\omega\tau$ radians!

Note that although the **delay** of device is a **constant** τ, the **phase shift** is a **function** of ω--in fact, it is directly proportional to frequency ω.

Note if the **input** signal for this device was of the form:

$$v_{in}(t) = \cos \omega t$$

Then the output would be:

$$\begin{aligned} v_{out}(t) &= \cos \omega (t - \tau) \\ &= \cos(\omega t - \omega \tau) \\ &= |H(\omega)| \cos(\omega t + \angle H(\omega)) \end{aligned}$$

Thus, we could **either** view the signal $v_{in}(t) = \cos \omega t$ as being delayed by an amount τ seconds, **or** phase shifted by an amount $-\omega\tau$ radians.

Q: *So, by* ***measuring*** *the output signal phase shift $\angle H(\omega)$, we could determine the delay τ through the device with the equation:*

$$\tau = -\frac{\angle H(\omega)}{\omega}$$

right?

A: Not exactly. The problem is that we cannot **unambiguously** determine the phase shift $\angle H(\omega) = -\omega\tau$ by **looking** at the output signal!

The reason is that $\cos(\omega t + \angle H(\omega))$ = $\cos(\omega t + \angle H(\omega) + 2\pi)$ = $\cos(\omega t + \angle H(\omega) - 4\pi)$, etc. More specifically:

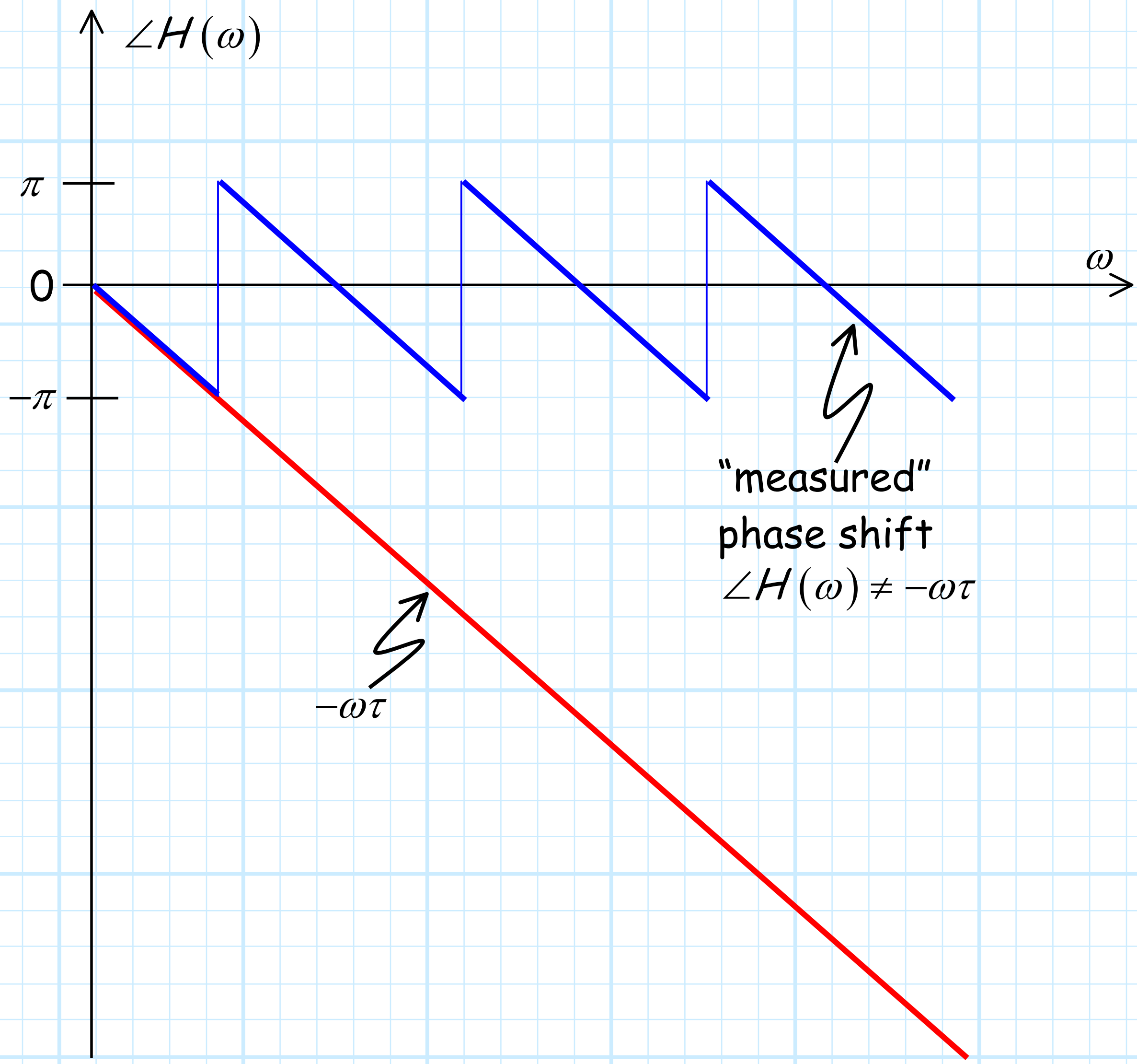
$$\cos(\omega t + \angle H(\omega)) = \cos(\omega t + \angle H(\omega) + n2\pi)$$

where n is **any integer**—positive or negative. We can't tell **which** of these output signal we are looking at!

Thus, any phase shift **measurement** has an inherent **ambiguity**. Typically, we interpret a phase measurement (in radians) such that:

$$-\pi < \angle H(\omega) \leq \pi \qquad \text{or} \qquad 0 \leq \angle H(\omega) < 2\pi$$

But almost certainly the actual value of $\angle H(\omega) = -\omega\tau$ is **nowhere** near these interpretations!



Clearly, using the equation:

$$\tau = -\frac{\angle H(\omega)}{\omega}$$

would **not** get us the correct result in this case—after all, there will be **several** frequencies ω with exactly the **same measured** phase $\angle H(\omega)$!

Q: *So determining the delay τ is **impossible**?*

A: NO! It is **entirely** possible—we simply must find the correct **method**.

Looking at the plot on the previous page, this method should become **apparent**. Not that although the measured phase (blue curve) is definitely **not** equal to the phase function $-\omega\tau$ (red curve), the **slope** of the two are **identical** at every point!

Q: *What good is knowing the **slope** of these functions?*

A: Just look! Recall that we can determine the slope by taking the first **derivative**:

$$\frac{\partial(-\omega\tau)}{\partial\omega} = -\tau$$

The slope directly tells us the **propagation delay**!

Thus, we can determine the propagation delay of this device by:

$$\tau = -\frac{\partial\angle H(\omega)}{\partial\omega}$$

where $\angle H(\omega)$ can be the **measured** phase. Of course, the method requires us to **measure** $\angle H(\omega)$ as a **function** of frequency (i.e., to make measurements at **many** signal frequencies).

Q: *__Now__ I see! If we wish to __determine__ the propagation delay τ through some __filter__, we simply need to take the derivative of $\angle S_{21}(\omega)$ with respect to frequency. __Right__?*

A: Well, sort of.

Recall for the **example** case that $h(t) = \delta(t-\tau)$ and $\angle H(\omega) = -\omega\tau$, where τ is a **constant**. For a microwave filter, **neither** of these conditions are true.

Specifically, the phase function $\angle S_{21}(\omega)$ will typically be some arbitrary **function** of frequency ($\angle S_{21}(\omega) \neq -\omega\tau$).

Q: *How could this be true? I thought you said that phase shift was __due__ to filter delay τ!*

A: Phase shift **is** due to device delay, it's just that the propagation delay of most devices (such as filters) is **not a constant**, but instead depends on the **frequency** of the signal propagating through it!

In other words, the propagation delay of a filter is typically some arbitrary **function** of frequency (i.e., $\tau(\omega)$). That's why the phase $\angle S_{21}(\omega)$ is **likewise** an arbitrary function of frequency.

Q: *Yikes! Is there __any__ way to determine the relationship between these two arbitrary functions?*

A: **Yes** there is! Just as before, the two can be related by a first **derivative**:

$$\tau(\omega) = -\frac{\partial \angle S_{21}(\omega)}{\partial \omega}$$

This result $\tau(\omega)$ is also know as **phase delay**, and is a **very** important function to consider when designing/specifying/ selecting a microwave **filter**.

Q: *Why; what might happen?*

A: If you get a filter with the wrong $\tau(\omega)$, your **output** signal could be horribly **distorted**—distorted by the evil effects of **signal dispersion**!